COHEN-MACAULAY CIRCULANT GRAPHS

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ABSTRACT. Let G be the circulant graph $C_n(S)$ with $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, and let $I(G)$ denote its edge ideal in the ring $R = k[x_1, \ldots, x_n]$. We consider the problem of determining when G is Cohen-Macaulay, i.e, $R/I(G)$ is a Cohen-Macaulay ring. Because a Cohen-Macaulay graph G must be well-covered, we focus on known families of well-covered circulant graphs of the form $C_n(1, 2, \ldots, d)$. We also characterize which cubic circulant graphs are Cohen-Macaulay. We end with the observation that even though the well-covered property is preserved under lexicographical products of graphs, this is not true of the Cohen-Macaulay property.

1. INTRODUCTION

Let $G = (V_G, E_G)$ denote a finite simple graph on the vertex set $V_G = \{x_1, \ldots, x_n\}$ with edge set E_G. By identifying the vertices of G with the variables of the polynomial ring $R = k[x_1, \ldots, x_n]$ (here, k is any field), we can associate to G the quadratic square-free monomial ideal

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E_G \rangle \subseteq R$$

called the *edge ideal* of G. Edge ideals were first introduced by Villarreal [17]. During the last couple of years, there has been an interest in determining which graphs G are *Cohen-Macaulay*, that is, determining when the ring $R/I(G)$ is a Cohen-Macaulay ring solely from the properties of the graphs. Although this problem is probably intractable for arbitrary graphs, results are known for some families of graphs, e.g., chordal graphs [8] and bipartite graphs [7]. Readers may also be interested in the recent survey of Morey and Villarreal [11] and the textbook of Herzog and Hibi [6], especially Chapter 9.

Our goal is to identify families of circulant graphs that are Cohen-Macaulay. Given an integer $n \geq 1$ and a subset $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, the *circulant graph* $C_n(S)$ is the graph on n vertices $\{x_1, \ldots, x_n\}$ such that $\{x_i, x_j\}$ is an edge of $C_n(S)$ if and only if $\min\{|i-j|, n-|i-j|\} \in S$. See, for example, the graph $C_{12}(1, 3, 4)$ in Figure 1. For convenience of notation, we suppress the set brackets for the set $S = \{1, 3, 4\}$ in $C_{12}(1, 3, 4)$. Circulant graphs belong to the family of Cayley graphs and are sometimes viewed as generalized cycles since $C_n = C_n(1)$. The complete graph is also a circulant graph because $K_n = C_n(1, 2, \ldots, \lfloor \frac{n}{2} \rfloor)$. In the literature, circulant graphs have appeared in a number of applications related to
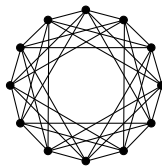
FIGURE 1. The circulant graph $C_{12}(1, 3, 4)$.

networks [1], error-correcting codes [14], and even music [3], in part, because of their regular structure (see [4]).

To classify families of Cohen-Macaulay circulant graphs we will use the fact that all Cohen-Macaulay graphs must be well-covered. A graph G is *well-covered* if all the maximal independent sets of G have the same cardinality, equivalently, every maximal independent set is a maximum independent set (see the survey of Plummer [12]). From an algebraic point-of-view, when a graph G is well-covered, the edge ideal is $I(G)$ is unmixed, that is, all of its associated primes have the same height. Some families of well-covered circulant graphs were recently classified by Brown and Hoshino [4]. Our main results (see Theorems 3.4 and 5.2) refine the work of Brown and Hoshino by determining which of these well-covered circulant graphs are also Cohen-Macaulay. In particular we show in Theorem 3.4 that for $n \geq 2d \geq 2$, the circulant $C_n(1, 2, \ldots, d)$ is Cohen-Macaulay if and only if $n \leq 3d + 2$ and $n \neq 2d + 2$. We also show that the Cohen-Macaulay graphs $C_n(1, 2, \ldots, d)$ are in fact vertex decomposable and shellable. Although the well-covered circulant graphs $C_{2d+2}(1, 2, \ldots, d)$ and $C_{4d+3}(1, 2, \ldots, d)$ are not Cohen-Macaulay, we prove that these graphs are Buchsbaum (see Theorem 3.7). We also classify which cubic circulant graphs are Cohen-Macaulay (see Theorem 5.5).

Our paper is structured as follows. In Section 2 we recall the relevant background regarding graph theory and simplicial complexes. In Section 3 we classify the Cohen-Macaulay graphs of the form $C_n(1, 2, \ldots, d)$ with $n \geq 2d$. Section 4 contains the proof of a lemma needed to prove the main result of Section 3. In Section 5, we look at cubic circulant graphs, and classify those that are Cohen-Macaulay. Section 6 contains some concluding comments and open questions related to the lexicographical product of graphs.

2. BACKGROUND DEFINITIONS AND RESULTS

A *simplicial complex* Δ on a vertex set $V = \{x_1, \ldots, x_n\}$ is a set of subsets of V that satisfies: (*i*) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$, and (*ii*) for each $i = 1, \ldots, n$, $\{x_i\} \in \Delta$. Note that condition (*i*) implies that $\emptyset \in \Delta$. The elements of Δ are called its *faces*. The maximal elements of Δ, with respect to inclusion, are the *facets* of Δ.

The *dimension* of a face $F \in \Delta$ is given by $\dim F = |F| - 1$; the *dimension* of a simplicial complex, denoted $\dim \Delta$, is the maximum dimension of all its faces. We call Δ a *pure* simplicial complex if all its facets have the same dimension. Let f_i be the number of faces of Δ of dimension i, with the convention that $f_{-1} = 1$. If $\dim \Delta = D$, then the *f-vector* of Δ is the $(D + 2)$-tuple $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_D)$. The *h-vector* of Δ is the

$(D+2)$-tuple $h(\Delta) = (h_0, h_1, \ldots, h_{D+1})$ with (see [18, Theorem 5.4.6])

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{D+1-i}{k-i} f_{i-1}.$$

Given any simplicial complex Δ on V, we can associate to Δ a monomial ideal I_Δ in the polynomial ring $R = k[x_1, \ldots, x_n]$ (with k a field) as follows:

$$I_\Delta = \langle \{x_{j_1} x_{j_2} \cdots x_{j_r} \mid \{x_{j_1}, \ldots, x_{j_r}\} \notin \Delta\} \rangle.$$

The ideal I_Δ is commonly called the *Stanley-Reisner ideal* of Δ, and the quotient ring R/I_Δ is the *Stanley-Reisner ring* of Δ.

We say that Δ is *Cohen-Macaulay* (over k) if its Stanley-Reisner ring R/I_Δ is a Cohen-Macaulay ring, that is, K-dim(R/I_Δ) = depth(R/I_Δ). Here K-dim(R/I_Δ), the *Krull dimension*, is the length of the longest chain of prime ideals in R/I_Δ with strict inclusions, and depth(R/I_Δ), the *depth*, is length of the longest sequence $f_1, \ldots, f_j$ in $\langle x_1, \ldots, x_n \rangle$ that forms a regular sequence on R/I_Δ.

We review the required background on reduced homology; see [10] for complete details. To any simplicial complex Δ with $f(\Delta) = (f_{-1}, f_0, \ldots, f_D)$ we can associate a reduced chain complex over k:

$$\tilde{C}.(\Delta; k) : 0 \longleftarrow k^{f_{-1}} \xleftarrow{\partial_0} k^{f_0} \xleftarrow{\partial_1} k^{f_1} \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_D} k^{f_D} \leftarrow 0.$$

Here k^{f_i} is the vector space with basis elements $e_{j_0, j_1, \ldots, j_i}$ corresponding to the i-dimensional faces of Δ. We assume $j_0 < j_1 < \cdots < j_i$. The boundary maps ∂_i are given by

$$\partial_i(e_{j_0, j_1, \ldots, j_i}) = e_{\hat{j}_0, j_1, \ldots, j_i} - e_{j_0, \hat{j}_1, \ldots, j_i} + e_{j_0, j_1, \hat{j}_2 \ldots, j_i} + \cdots + (-1)^i e_{j_0, j_1, \ldots, \hat{j}_i}$$

where $\hat{\ }$ denotes an omitted term. The *ith reduced simplicial homology* of Δ with coefficients in k is the k-vector space

$$\tilde{H}_i(\Delta; k) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1}).$$

The dimensions of $\tilde{H}_i(\Delta; k)$ are related to $f(\Delta)$ via the *reduced Euler characteristic*:

(2.1) $$\sum_{i=-1}^{D} (-1)^i \dim_k \tilde{H}_i(\Delta; k) = \sum_{i=-1}^{D} (-1)^i f_i.$$

We will find it convenient to use Reisner's Criterion. Given a face $F \in \Delta$, the *link* of F in Δ is the subcomplex

$$\operatorname{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset, \ F \cup G \in \Delta\}.$$

Theorem 2.1 (Reisner's Criterion). *Let Δ be a simplicial complex on V. Then R/I_Δ is Cohen-Macaulay over k if and only if for all $F \in \Delta$, $\tilde{H}_i(\operatorname{link}_\Delta(F); k) = 0$ for all $i < \dim \operatorname{link}_\Delta(F)$.*

For any vertex $x \in V$, the *deletion* of x in Δ is the subcomplex

$$\operatorname{del}_\Delta(\{x\}) = \{F \in \Delta \mid x \notin F\}.$$

The following combinatorial topology property was introduced by Provan and Billera [13].

Definition 2.2. Let Δ be a pure simplicial complex. Then Δ is *vertex decomposable* if

    (i) Δ is a simplex, i.e $\{x_1, \ldots, x_n\}$ is the unique maximal facet, or
    (ii) there exists an $x \in V$ such that $\mathrm{link}_\Delta(\{x\})$ and $\mathrm{del}_\Delta(\{x\})$ are vertex decomposable.

We will also refer to the following family of simplicial complexes.

Definition 2.3. Let Δ be a pure simplicial complex with facets $\{F_1, \ldots, F_t\}$. Then Δ is *shellable* if there exists an ordering of $F_1, \ldots, F_t$ such that such that for all $1 \leq j < i \leq t$, there is some $x \in F_i \setminus F_j$ and some $k \in \{1, \ldots, j-1\}$ such that $\{x\} = F_i \setminus F_k$.

The following theorem summarizes a number of necessary and sufficient conditions of Cohen-Macaulay simplicial complexes that we require.

Theorem 2.4. *Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$.*

    (i) *If Δ is Cohen-Macaulay, then it is pure.*
    (ii) *If Δ is Cohen-Macaulay, then $h(\Delta)$ has only non-negative entries.*
    (iii) *If $n - \mathrm{pdim}(R/I_\Delta) = \mathrm{K\text{-}dim}(R/I_\Delta)$, then Δ is Cohen-Macaulay (here, $\mathrm{pdim}(R/I_\Delta)$ denotes the projective dimension of R/I_Δ, the length of a minimal free resolution of R/I_Δ).*
    (iv) *If Δ is vertex decomposable, then Δ is Cohen-Macaulay.*
    (v) *If $\dim \Delta = 0$, then Δ is vertex decomposable/shellable/Cohen-Macaulay.*
    (vi) *If $\dim \Delta = 1$, then Δ is vertex decomposable/shellable/Cohen-Macaulay if and only if Δ is connected.*

Proof. Many of these results are standard. For (i) see [18, Theorem 5.3.12]; for (ii) see [18, Theorem 5.4.8]; (iii) follows from the Auslander-Buchsbaum Theorem; for (iv) see [13, Corollary 2.9] and the fact that shellable implies Cohen-Macaulay [18, Theorem 5.3.18]; (v) is [13, Proposition 3.1.1]; and (vi) is [13, Theorem 3.1.2]. $\square$

In this paper, we will be interested in independence complexes of finite simple graphs $G = (V_G, E_G)$. We say that a set of vertices $W \subseteq V_G$ is an *independent set* if for all $e \in E_G$, $e \not\subseteq W$. The *independence complex* of G is the set of all independent sets:

$$\mathrm{Ind}(G) = \{W \mid W \text{ is an independent set of } V_G\}.$$

The set $\mathrm{Ind}(G)$ is a simplicial complex. Following convention, G is Cohen-Macaulay (resp. shellable, vertex decomposable) if $\mathrm{Ind}(G)$ is Cohen-Macaulay (resp. shellable, vertex decomposable).

The facets of $\mathrm{Ind}(G)$ correspond to the *maximal independent sets* of vertices of G. It is common to let $\alpha(G)$ denote the cardinality of a maximum independent set of vertices in G. A graph G is *well-covered* if every maximal independent set has cardinality $\alpha(G)$. Moreover, a direct translation of the definitions gives us:

Lemma 2.5. *If G is Cohen-Macaulay, then G is well-covered.*

3. CHARACTERIZATION OF CIRCULANT GRAPHS $C_n(1, 2, \ldots, d)$.

In this section, we classify which circulant graphs of the form $C_n(1, 2, \ldots, d)$ are Cohen-Macaulay. Brown and Hoshino recently classified the well-covered graphs in this family:

Theorem 3.1. ([4, Theorem 4.1]). *Let n and d be integers with $n \geq 2d \geq 2$. Then $C_n(1, 2, \ldots, d)$ is well-covered if and only if $n \leq 3d + 2$ or $n = 4d + 3$.*

Brown and Hoshino's result is a key ingredient for our main result. We also need one additional result of [4] on the independence polynomial of $C_n(1, 2, \ldots, d)$, but translated into a statement about f-vectors. The *independence polynomial* of a graph G is given by $I(G, x) = \sum_{k=0}^{\alpha(G)} i_k x^k$ where i_k is the number of independent sets of cardinality k (we take $i_0 = 1$). Note that if $\Delta = \text{Ind}(G)$ and $f(\Delta) = (f_{-1}, f_0, \ldots, f_D)$, then $i_k = f_{k-1}$ for each k. If we translate [4, Theorem 3.1] into the language of f-vectors and independence complexes, we get the following statement.

Lemma 3.2. *Let n and d be integers with $n \geq 2d \geq 2$, $G = C_n(1, 2, \ldots, d)$, and $D = \dim \text{Ind}(G)$. Then $D = \lfloor \frac{n}{d+1} \rfloor - 1$ and $f(\Delta) = (f_{-1}, f_0, \ldots, f_D)$ where*

$$f_{k-1} = \frac{n}{n - dk} \binom{n - dk}{k} \text{ for } k = 0, \ldots, (D + 1).$$

By Lemma 2.5, to characterize the Cohen-Macaulay circulant graphs of the form $C_n(1, 2, \ldots, d)$, it suffices to determine which of the well-covered graphs of Theorem 3.1 are also Cohen-Macaulay. Interestingly, proving that $C_n(1, 2, \ldots, d)$ is *not* Cohen-Macaulay when $n = 4d + 3$ is the most subtle part of this proof. To carry out this part of the proof, we need the following lemma, whose proof we postpone until the next section.

Lemma 3.3. *Fix an integer $d \geq 3$, and let $G = C_{4d+3}(1, 2, \ldots, d)$. If $\Delta = \text{Ind}(G)$, then*

$$\dim_k \tilde{H}_2(\Delta; k) \geq \frac{(4d + 3)}{3} \binom{d - 1}{2}.$$

Assuming, for the moment, that Lemma 3.3 holds, we arrive at our main result:

Theorem 3.4. *Let n and d be integers with $n \geq 2d \geq 2$ and let $G = C_n(1, 2, \ldots, d)$. Then the following are equivalent:*

(i) G is Cohen-Macaulay.
(ii) G is shellable.
(iii) G is vertex decomposable.
(iv) $n \leq 3d + 2$ and $n \neq 2d + 2$.

Proof. We always have $(iii) \Rightarrow (ii) \Rightarrow (i)$. We now prove that $(iv) \Rightarrow (iii)$.

By Lemma 3.2, when $n = 2d$ or $n = 2d + 1$, $\dim \text{Ind}(G) = 0$. Now apply Theorem 2.4 (v).

When $2d + 2 \leq n \leq 3d + 2$, $\dim \text{Ind}(G) = \lfloor \frac{n}{d+1} \rfloor - 1 = 1$. Let $V = \{x_1, \ldots, x_n\}$. If $n = 2d + 2$, then $\text{Ind}(G)$ is not connected, because the only edges of $\text{Ind}(G)$ are $\{x_i, x_{d+1+i}\}$ for $i = 1, \ldots, d + 1$. On the other hand, when $2d + 3 \leq n \leq 3d + 2$, $\text{Ind}(G)$ is connected.

To see this, let $n = 2d + c$ for $3 \leq c \leq d + 2$. For each $i = 1, 2, \ldots, n$, $\{x_i, x_{i+d+2}\}$ and $\{x_{i+1}, x_{i+d+2}\} \in \text{Ind}(G)$, with subscript addition adjusted modulo n. Thus, for any $x_i, x_j \in V$ with $i < j$, we can make the path $x_i, x_{i+d+2}, x_{i+1}, x_{i+d+3}, x_{i+2}, \ldots, x_j$. So, $\text{Ind}(G)$ is connected. Applying Theorem 2.4 (vi) then shows that $(iv) \Rightarrow (iii)$.

To complete the proof, we will show that if $n \geq 2d$ with $n = 2d + 2$ or $n > 3d + 2$, then G is not Cohen-Macaulay. In the proof that $(iv) \Rightarrow (iii)$, we already showed that if $n = 2d + 2$, then $\text{Ind}(G)$ is not connected and $\dim \text{Ind}(G) = 1$. Again by Theorem 2.4 (vi) this implies G is not Cohen-Macaulay.

If $n > 3d + 2$ and $n \neq 4d + 3$, then by Theorem 3.1, G is not well-covered, and consequently, by Lemma 2.5, G is not Cohen-Macaulay. It therefore remains to show that if $n = 4d + 3$, then G is not Cohen-Macaulay for all $d \geq 1$. The remainder of this proof is dedicated to this case.

By Lemma 3.2, $\dim \text{Ind}(G) = 2$ and the f-vector of $\text{Ind}(G)$ is given by

$$f(\text{Ind}(G)) = \left(1, 4d + 3, 4d^2 + 7d + 3, \frac{4d^3 + 15d^2 + 17d + 6}{6}\right).$$

When $d = 1$, then $f(\text{Ind}(G)) = (1, 7, 14, 7)$ and hence $h(\text{Ind}(G)) = (1, 4, 3, -1)$. When $d = 2$, then $f(\text{Ind}(G)) = (1, 11, 33, 21)$ and hence $h(\text{Ind}(G)) = (1, 8, 13, -1)$. In these two cases, Theorem 2.4 (ii) implies G is not Cohen-Macaulay.

We can therefore assume that $d \geq 3$. To show that $\text{Ind}(G)$ is not Cohen-Macaulay, we will show that $\tilde{H}_1(\text{Ind}(G); k) \neq 0$. This suffices because $\text{Ind}(G) = \text{link}_{\text{Ind}(G)}(\emptyset)$, so Reisner's Criterion (Theorem 2.1) would imply that $\text{Ind}(G)$ is not Cohen-Macaulay.

Using the fact that $\dim \text{Ind}(G) = 2$, the f-vector given above, and the reduced Euler characteristic (2.1) we know

$$-1 + (4d + 3) - (4d^2 + 7d + 3) + \frac{4d^3 + 15d^2 + 17d + 6}{6} = \sum_{i=-1}^{2} (-1)^i \dim_k \tilde{H}_i(\text{Ind}(G); k).$$

Because $\text{Ind}(G)$ is a non-empty connected simplicial complex, we have $\dim_k \tilde{H}_i(\text{Ind}(G); k) = 0$, for $i = -1$, and 0. Simplifying both sides of the above equation and rearranging gives:

$$\dim_k \tilde{H}_1(\text{Ind}(G); k) = \dim_k \tilde{H}_2(\text{Ind}(G); k) - \frac{d(4d^2 - 9d - 1)}{6}.$$

By Lemma 3.3

$$\dim_k \tilde{H}_1(\text{Ind}(G); k) \geq \frac{4d + 3}{3} \binom{d - 1}{2} - \frac{d(4d^2 - 9d - 1)}{6} = 1.$$

So, $\tilde{H}_1(\text{Ind}(G); k) \neq 0$ as desired. $\qquad\square$

When we specialize the above theorem to the case $d = 1$, we recover the known classification of the Cohen-Macaulay cycles [18, Corollary 6.3.6]. Note that $C_2(1) = K_2$ is also Cohen-Macaulay, but it is not a cycle.

Corollary 3.5. *Let $n \geq 3$. Then $C_n = C_n(1)$ is Cohen-Macaulay if and only if $n = 3$ or 5.*

Even though $C_{2d+2}(1, 2, \ldots, d)$ and $C_{4d+3}(1, 2, \ldots, d)$ are not Cohen-Macaulay, they still have an interesting algebraic structure, as noted in Theorem 3.7 below.

Definition 3.6. A pure simplicial complex Δ is called *Buchsbaum* over a field k if for every non-empty face $F \in \Delta$, $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$ for all $i < \dim \mathrm{link}_\Delta(F)$. We say a graph G is *Buchsbaum* if the independence complex of G is Buchsbaum.

Note that by Reisner's Criterion (Theorem 2.1), if G is Cohen-Macaulay, then G is Buchsbaum. We can now classify all circulant graphs of the form $C_n(1, \ldots, d)$ which are Buchsbaum, but not Cohen-Macaulay.

Theorem 3.7. *Let n and d be integers with $n \geq 2d$ and $d \geq 1$. Let $G = C_n(1, 2, \ldots, d)$. Then G is Buchsbaum, but not Cohen-Macaulay if and only if $n = 2d + 2$ or $n = 4d + 3$.*

Proof. ($\Rightarrow$) For G to be Buchsbaum, $\mathrm{Ind}(G)$ must be pure, that is G is well-covered. By Theorem 3.1, $2d \leq n \leq 3d + 2$ or $n = 4d + 3$. Because G is not Cohen-Macaulay, Theorem 3.4 implies $n = 2d + 2$ or $n = 4d + 3$.

($\Leftarrow$) We first show that if $n = 4d + 3$, then G is Buchsbaum. Let $\Delta = \mathrm{Ind}(G)$. Since $\dim \Delta = 2$, given any $F \in \Delta$, $|F| \in \{0, 1, 2, 3\}$. We wish to show that if $|F| = 1, 2$, or 3, then $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$ for all $i < \dim \mathrm{link}_\Delta(F)$.

If $|F| = 3$, then $\mathrm{link}_\Delta(F) = \{\emptyset\}$, and hence $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$ for all $i < \dim \mathrm{link}_\Delta(F) = -1$. When $|F| = 2$, then $\dim \mathrm{link}_\Delta(F) = 0$, and again, we have $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$ for all $i < \dim \mathrm{link}_\Delta(F) = 0$.

It therefore suffices to show that when $|F| = 1$, then $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$ for all $i < \dim \mathrm{link}_\Delta(F)$. Because of the symmetry of G, we can assume without a loss of generality that $F = \{x_1\}$. Because G is well-covered, any independent set containing x_1 can be extended to a maximal independent set, and furthermore, this independent set has cardinality three. This, in turn, implies that $\dim \mathrm{link}_\Delta(F) = 1$. For any $i < 0$, $\tilde{H}_i(\mathrm{link}_\Delta(F); k) = 0$, so it suffices to prove that $\tilde{H}_0(\mathrm{link}_\Delta(F); k) = 0$. Proving this condition is equivalent to proving that $\mathrm{link}_\Delta(F)$ is connected.

We first note that none of the vertices $x_2, x_3, \ldots, x_{d+1}, x_{3d+4}, x_{3d+5}, \ldots, x_{4d+3}$ appear in $\mathrm{link}_\Delta(\{x_1\})$ because these vertices are all adjacent to x_1 in G. On the other hand, the following elements are facets of Δ:

$$\{x_1, x_{d+2}, x_{2d+3}\}, \{x_1, x_{d+2}, x_{2d+4}\}, \ldots, \{x_1, x_{d+2}, x_{3d+3}\},$$
$$\{x_1, x_{d+3}, x_{3d+3}\}, \{x_1, x_{d+4}, x_{3d+3}\}, \ldots, \{x_1, x_{2d+2}, x_{3d+3}\}.$$

Consequently the following edges are in $\mathrm{link}_\Delta(\{x_1\})$:

$$\{x_{d+2}, x_{2d+3}\}, \{x_{d+2}, x_{2d+4}\}, \ldots, \{x_{d+2}, x_{3d+3}\}, \{x_{d+3}, x_{3d+3}\}, \{x_{d+4}, x_{3d+3}\}, \ldots, \{x_{2d+2}, x_{3d+3}\}.$$

Thus $\mathrm{link}_\Delta(\{x_1\})$ is connected, as desired.

Now suppose $n = 2d + 2$. As shown in the proof of Theorem 3.4, $\mathrm{Ind}(G)$ consists of the disjoint edges $\{x_i, x_{d+1+i}\}$ for $i = 1, \ldots, d + 1$. If $F \in \mathrm{Ind}(G)$ and $|F| = 2$, then $\mathrm{link}_\Delta(F) = \{\emptyset\}$. If $F \in \mathrm{Ind}(G)$ and $|F| = 1$, then $\mathrm{link}_\Delta(F) = \{\{x\}\}$ for some variable x. Therefore G is Buchsbaum. $\square$

4. Proof of Lemma 3.3

The purpose of this section is to prove Lemma 3.3. We will be interested in finding induced octahedrons in our independence complex.

Lemma 4.1. *Fix an integer $d \geq 3$. Let $G = C_{4d+3}(1, 2, \ldots, d)$ and let $\Delta = \mathrm{Ind}(G)$ be the associated independence complex. Let $W = \{i_1, i_2, j_1, j_2, k_1, k_2\} \subseteq V_G$ be six distinct vertices. Then the induced simplicial complex $\Delta|_W = \{F \in \Delta \mid F \subseteq W\}$ is isomorphic to the labeled octahedron in Figure 2 if and only if the induced graph G_W is the graph of three disjoint edges $\{i_1, i_2\}$, $\{j_1, j_2\}$, and $\{k_1, k_2\}$.*
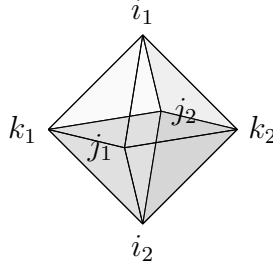

FIGURE 2. A labeled octahedron

Proof. Suppose that $\Delta|_W$ is isomorphic to the octahedron in Figure 2. It follows that $\{i_1, i_2\}$, $\{j_1, j_2\}$ and $\{k_1, k_2\}$, which are not edges of the octahedron, are also not edges of Δ. Because Δ is an independence complex, these means that each set is not an independent set, or in other words, $e_1 = \{i_1, i_2\}$, $e_2 = \{j_1, j_2\}$, and $e_3 = \{k_1, k_2\}$ are all edges of G. It suffices to show that G_W consists only of these edges. If not, there is a vertex $x \in e_i$ and a vertex $y \in e_j$ with $i \neq j$, such that $\{x, y\}$ is an edge of G. However, for any $x \in e_i$ and $y \in e_j$, $\{x, y\}$ is an edge of $\Delta|_W$, and consequently, $\{x, y\}$ cannot be an edge of G, a contradiction.

For the converse, we reverse the argument. If G_W is the three disjoint edges $\{i_1, i_2\}$, $\{j_1, j_2\}$ and $\{k_1, k_2\}$, then it follows that $\{i_1, j_1, k_1\}$, $\{i_1, j_1, k_2\}$, $\{i_1, j_2, k_1\}$, $\{i_1, j_2, k_2\}$, $\{i_2, j_1, k_1\}$, $\{i_2, j_1, k_2\}$, $\{i_2, j_2, k_1\}$, $\{i_2, j_2, k_2\}$ are all independent sets of G, and thus belong to Δ, and consequently, $\Delta|_W$. Because $\{i_1, i_2\}$, $\{j_1, j_2\}$, and $\{k_1, k_2\}$ are not faces of Δ, the facets of the complex $\Delta|_W$ are these eight faces, whence $\Delta|_W$ is an octahedron. $\square$

We now come to our desired proof.

Proof. (of Lemma 3.3) We begin by first recalling some facts about $\Delta = \mathrm{Ind}(G)$. By Theorem 3.1 and Lemma 3.2, the simplicial complex Δ is pure and two dimensional with $f(\Delta) = (f_{-1}, f_0, f_1, f_2)$. Therefore, the reduced chain complex of Δ over k has the form

$$0 \longleftarrow k^{f_{-1}} \xleftarrow{\partial_0} k^{f_0} \xleftarrow{\partial_1} k^{f_1} \xleftarrow{\partial_2} k^{f_2} \longleftarrow 0.$$

It follows from this chain complex that $\dim_k \tilde{H}_2(\Delta; k) = \dim_k \ker \partial_2$.

Our strategy, therefore, is to identify $\frac{(4d+3)}{3}\binom{d-1}{2}$ linearly independent elements in $\ker \partial_2$. Note that if $W \subseteq V$ is a subset of the vertices such that the induced complex $\Delta|_W$ is isomorphic to an octahedron, then this octahedron corresponds to an element of $\ker \partial_2$. We make this more precise. Suppose that $W = \{i_1, i_2, j_1, j_2, k_1, k_2\} \subseteq V$ and $\Delta|_W$ is an octahedron, i.e, the simplicial complex with facets

$$
\begin{aligned}
\Delta|_W \;=\; & \langle \{i_1, j_1, k_1\}, \{i_1, j_1, k_2\}, \{i_1, j_2, k_1\}, \{i_1, j_2, k_2\}, \\
& \{i_2, j_1, k_1\}, \{i_2, j_1, k_2\}, \{i_2, j_2, k_1\}, \{i_2, j_2, k_2\} \rangle.
\end{aligned}
$$

Note that each $\{i_a, j_b, k_c\}$ is a 2-dimensional face of Δ; we associate to $\Delta|_W$ the following element of k^{f_2}:

$$
O_W = e_{i_1 j_1 k_1} - e_{i_1 j_1 k_2} - e_{i_1 j_2 k_1} - e_{i_2 j_1 k_1} + e_{i_1 j_2 k_2} + e_{i_2 j_1 k_2} + e_{i_2 j_2 k_1} - e_{i_2 j_2 k_2}.
$$

Here, we have assumed that the indices of each basis element have been written in increasing order. The boundary map ∂_2 evaluated at O_W gives $\partial_2(O_W) = 0$, i.e., $O_W \in \ker \partial_2$.

To compute the lower bound on $\dim_k \tilde{H}_2(\Delta; k)$, we will build a list L of octahedrons in Δ and then order the elements of L using the lexicographical ordering so that each octahedron in the list L contains a face that has not appeared in any previous octahedron in L with respect to the ordering. By associating each octahedron to the corresponding element of k^{f_2}, each octahedron will belong to $\ker \partial_2$. Moreover, the fact that each octahedron in L has a face that has not appeared previously implies that the octahedron can not be written as a linear combination of our previous elements in $\ker \partial_2$, thus giving us the required number of linearly independent elements.

By Lemma 4.1, there is a one-to-one correspondence between the induced octahedrons of Δ and the induced subgraphs of G consisting of three pairwise disjoint edges. So we can represent an octahedron by a tuple $(i_1, i_2; j_1, j_2; k_1, k_2)$ where $\{i_1, i_2\}$, $\{j_1, j_2\}$, and $\{k_1, k_2\}$ correspond to these edges.

We begin by considering the octahedrons described by the following list:

(4.1)

$$
\begin{array}{llll}
(1, 2; & d+3, d+4; & 2d+5, 3d+3) \\
(1, 2; & d+3, d+4; & 2d+6, 3d+3) \\
& & \vdots \\ \hline
(1, 2; & d+3, d+4; & 3d+2, 3d+3) \\ \hline
(1, 2; & d+3, d+5; & 2d+6, 3d+3) \\
(1, 2; & d+3, d+5; & 2d+7, 3d+3) \\
& & \vdots \\ \hline
(1, 2; & d+3, d+5; & 3d+2, 3d+3) \\ \hline
(1, 2; & d+3, d+6; & 2d+7, 3d+3) \\
& & \vdots \\ \hline
(1, 2; & d+3, 2d; & 3d+1, 3d+3) \\ \hline
(1, 2; & d+3, 2d; & 3d+2, 3d+3) \\ \hline
(1, 2; & d+3, 2d+1; & 3d+2, 3d+3)
\end{array}
$$

If we take our list of octahedrons in (4.1) and add one to each index, we will get a new list of octahedrons. In terms of the graph $G = C_{4d+3}(1, 2, \ldots, d)$, we are "rotating" our disjoint

edges to the right. We "rotate" these disjoint edges, or equivalently, we add one to each index, until $k_1 = 4d+3$. So, for example, the disjoint edges $(1, 2; d+3, d+4; 2d+5, 3d+3)$ can be rotated to the right $2d - 2$ times to give us $2d - 1$ octahedrons

$$
\begin{array}{lll}
(1, 2; & d + 3, d + 4; & 2d + 5, 3d + 3) \\
(2, 3; & d + 4, d + 5; & 2d + 6, 3d + 4) \\
(3, 4; & d + 5, d + 6; & 2d + 7, 3d + 5) \\
\quad\vdots & \quad\vdots & \quad\vdots \\
(2d - 1, 2d; & 3d + 1, 3d + 2; & 4d + 3, d - 2).
\end{array}
$$

On the other hand, the disjoint edges $(1, 2; d + 3, 2d + 1; 3d + 2, 3d + 3)$ are only rotated $d + 1$ times to create $d + 2$ octahedrons

$$
\begin{array}{lll}
(1, 2; & d + 3, 2d + 1; & 3d + 2, 3d + 3) \\
(2, 3; & d + 4, 2d + 2; & 3d + 3, 3d + 4) \\
(3, 4; & d + 5, 2d + 3; & 3d + 3, 3d + 5) \\
\quad\vdots & \quad\vdots & \quad\vdots \\
(d + 2, d + 3; & 2d + 4, 3d + 2; & 4d + 3, 1) \qquad .
\end{array}
$$

If we carry out this procedure, we end up with an expanded list L of octahedrons with

$$
|L| = \sum_{k=1}^{d-2} k(2d - k) = 2d \sum_{k=1}^{d-2} k - \sum_{k=1}^{d-2} k^2 = \frac{4d + 3}{3} \binom{d - 1}{2}.
$$

To see why, there is only one collection of disjoint edges with $k_1 = 2d+5$ which is rotated $2d - 1$ times, there are two tuples of disjoint edges with $k_1 = 2d + 6$ which are rotated $2d - 2$ times, and so on, until we arrive at the $d - 2$ tuples which are constructed from all the tuples with $k_1 = 3d + 2$ rotated $d + 2$ times.

It now suffices to show that the corresponding elements of $\ker \partial_2$ are linearly independent. In (4.2), we have arranged the list L in lexicographical order from smallest to largest:

(4.2)
$$
\begin{array}{lll}
(1, 2; & d + 3, \ d + 4; & 2d + 5, 3d + 3) \\
& \quad\vdots & \quad\vdots \\
(1, 2; & d + 3, 2d + 1; & 3d + 2, 3d + 3) \\
\hline
(2, 3; & d + 4, d + 5; & 2d + 6, 3d + 4) \\
& \quad\vdots & \quad\vdots \\
(2, 3; & d + 4, 2d + 2; & 3d + 3, 3d + 4) \\
\hline
& \quad\vdots & \\
\hline
(2d - 2, 2d - 1; & 3d, 3d + 1; & 4d + 3, d - 3) \\
(2d - 2, 2d - 1; & 3d, 3d + 2; & 4d + 3, d - 3) \\
(2d - 1, 2d; & 3d + 1, 3d + 2; & 4d + 3, d - 2)
\end{array}
$$

For each $(i_1, i_2; j_1, j_2; k_1, k_2)$ in L, we consider the two-dimensional face $\{i_2, j_2, k_1\}$ of the associated octahedron. We claim that as we progress down the list in (4.2), each face $\{i_2, j_2, k_1\}$ has not appeared in a previous octahedron.

In particular, suppose that $(i_1, i_2; j_1, j_2; k_1, k_2)$ is the ℓ-th item in (4.2). We wish to show that the face $\{i_2, j_2, k_1\}$ has not appeared in any of the first $\ell - 1$ octahedrons in the lexicographically ordered list (4.2). Suppose, that $(a_1, a_2; b_1, b_2, c_1, c_2)$ appears earlier in the list and contains the face $\{i_2, j_2, k_1\}$. For this face to appear, $\{a_1, a_2\}$ must contain exactly one of i_2, j_2, k_1, $\{b_1, b_2\}$ must contain exactly one of the remaining two vertices, and $\{c_1, c_2\}$, must contain the remaining vertex of the face.

By the way we listed and constructed our octahedrons, $i_2 < j_2 < k_1$, $i_1 = i_2 - 1$, and $j_1 = i_2 + d + 1$. Further, $k_2 = i_2 + 3d + 1$ if $i_2 + 3d + 1 \leq (4d - 3)$, and $k_2 = i_2 + 3d + 1 - (4d - 3)$ if $i_2 + 3d + 1 > (4d - 3)$. Note that $a_1 \neq i_2, j_2$ or k_2 otherwise $a_1 > i_1$, contradicting the lexicographical ordering. Also, if $a_2 = j_2$, then $a_1 = j_2 - 1 \geq i_2 > i_1$, again a contradiction. The same problem arises if $a_2 = k_1$. Thus $\{a_1, a_2\} = \{i_2 - 1, i_2\} = \{i_1, i_2\}$, i.e., $(a_1, a_2; b_1, b_2; c_1, c_2) = (i_1, i_2, j_1, b_2, c_1, k_2)$.

Since j_2 and k_1 must also appear in this tuple, there are only two possibilities:

$$(a_1, a_2; b_1, b_2; c_1, c_2) = (i_1, i_2, j_1, k_1, j_2, k_2) \text{ or } (i_1, i_2, j_1, j_2, k_1, k_2).$$

But neither of these tuples appear strictly before $(i_1, i_2, j_1, j_2, k_1, k_2)$ with respect to our ordering, thus completing the proof. $\square$

5. Cohen-Macaulay Circulant Cubic Graphs

Brown and Hoshino [4] classified which circulant cubic graphs are well-covered. Recall that a *cubic* graph is a graph in which each vertex has degree 3. Thus, if G is a circulant cubic graph, then $G = C_{2n}(a, n)$ for some $1 \leq a < n$.

There are only a finite number of connected well-covered circulant cubic graphs:

Theorem 5.1 ([4, Theorem 4.3])**.** *Let G be a connected circulant cubic graph. Then G is well-covered if and only if it is isomorphic to one of the following graphs: $C_4(1, 2)$, $C_6(1, 3)$, $C_6(2, 3)$, $C_8(1, 4)$ or $C_{10}(2, 5)$.*

Using a computer algebra system like *Macaulay2* [9], one can simply check which of these graphs, displayed in Figure 3, are Cohen-Macaulay.

Theorem 5.2. *Let G be a connected circulant cubic graph. Then is Cohen-Macaulay if and only if it is isomorphic to $C_4(1, 2)$ or $C_6(2, 3)$.*

Proof. By Theorem 5.1, it suffices to check which of the graphs $C_4(1, 2)$, $C_6(1, 3)$, $C_6(2, 3)$, $C_8(1, 4)$ or $C_{10}(2, 5)$, are also Cohen-Macaulay.
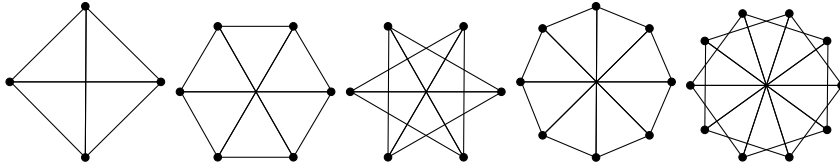

FIGURE 3. The well-covered connected cubic circulant graphs.

For any graph G, $\dim R/I(G) = \alpha(G)$. So, by Theorem 2.4 (iii), we simply need to check if $\alpha(G) = n - \text{pdim}(R/I(G))$. We can compute $\alpha(G)$ for each of the graphs G in Figure 3 by inspection; on the other hand, we compute the projective dimension using a computer algebra system. The following table summarizes these calculations:

G	$C_4(1,2)$	$C_6(1,3)$	$C_6(2,3)$	$C_8(1,4)$	$C_{10}(2,5)$
$n - \text{pdim}(R/I(G))$	1	1	2	2	2
$\alpha(G)$	1	3	2	3	4

The conclusion now follows from the values in the table. $\qquad\square$

As in Brown and Hoshino [4], we will use the following result to extend Theorem 5.2 to all circulant cubic graphs. The following classification is due to Davis and Domke [5].

Theorem 5.3. *Let $G = C_{2n}(a, n)$ with $1 \le a < n$, and let $t = \gcd(a, 2n)$.*

(i) *If $\frac{2n}{t}$ is even, then G is isomorphic to t copies of $C_{\frac{2n}{t}}(1, \frac{n}{t})$.*

(ii) *If $\frac{2n}{t}$ is odd, then G is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t}}(2, \frac{2n}{t})$.*

We also use the following lemma in the next proof.

Lemma 5.4 ([18, Proposition 6.2.8]). *Suppose that the graph $G = H \cup K$ where H and K are disjoint components of G. Then G is Cohen-Macaulay if and only if H and K are Cohen-Macaulay.*

Theorem 5.5. *Let $G = C_{2n}(a, n)$ with $1 \le a < n$, that is, G is a cubic circulant graph. Let $t = \gcd(a, 2n)$. Then G is Cohen-Macaulay if and only if $\frac{2n}{t} = 3$ or 4.*

Proof. Suppose that $\frac{2n}{t} \ne 3$ or 4. If $\frac{2n}{t}$ is even, then $C_{\frac{2n}{t}}(1, \frac{n}{t})$ is not Cohen-Macaulay by Theorem 5.2 and if $\frac{2n}{t}$ is odd, then $C_{\frac{4n}{t}}(2, \frac{2n}{t})$ is also not Cohen-Macaulay by Theorem 5.2. Thus by Lemma 5.4 G is not Cohen-Macaulay. Conversely, if $\frac{2n}{t} = 4$, then by Theorem 5.3, G is isomorphic to t copies of $C_4(1,2)$ and if $\frac{2n}{t} = 3$, then G is isomorphic to $\frac{t}{2}$ copies of $C_6(2,3)$. In both cases, Theorem 5.2 and Lemma 5.4 imply G is Cohen-Macaulay. $\square$

6. CONCLUDING COMMENTS AND OPEN QUESTIONS

The question of classifying *all* Cohen-Macaulay circulant graphs $C_n(S)$ is probably an intractable problem. Even the weaker question of determining whether or not a circulant graph $G_n(S)$ is well-covered (equivalently, $\text{Ind}(C_n(S))$ is a pure simplicial complex) was shown by Brown and Hoshino to be co-NP-complete [4, Theorem 2.5]. At present, the best we can probably expect is to identify families of Cohen-Macaulay circulant graphs.

Brown and Hoshino observed that circulant graphs behave well with respect to the lexicographical product. Recall this construction:

Definition 6.1. Given two graphs G and H, the *lexicographical product*, denoted $G[H]$, is graph with vertex set $V(G) \times V(H)$, where any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either $\{u, x\} \in G$ or $u = x$ and $\{v, y\} \in H$.

When G and H are both circulant graphs, then the lexicographical product $G[H]$ is also circulant (see [4, Theorem 4.6]). The well-covered property is also preserved with respect to the lexicographical product (see [16]).

Theorem 6.2. *Let G and H be two non-empty graphs. Then $G[H]$ is well-covered if and only if the graphs G and H are well-covered.*

As a consequence, the families of well-covered circulant graphs discovered in [4] can be combined into new well-covered circulant graphs using the lexicographical product. It is therefore natural to ask if the lexicographical product allows us to build new Cohen-Macaulay circulant graphs from known Cohen-Macaulay circulant graphs. In other words, can we replace "well-covered" in Theorem 6.2 by "Cohen-Macaulay". This turns out not to always be the case, as the following example shows.

Example 6.3. Let G and H be the Cohen-Macaulay circulant graphs $G = C_2(1)$ and $H = C_5(1)$. Then $G[H] = C_{10}(1, 4, 5)$ and $H[G] = C_{10}(1, 2, 3, 5)$ as seen in Figure 4. We can compute $\alpha(G)$ for the graphs in Figure 4 by inspection; on the other hand,
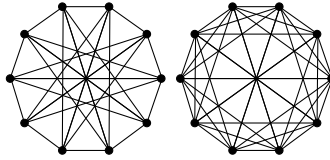

FIGURE 4. Lexicographical products $C_2[C_5]$ and $C_5[C_2]$

we compute the projective dimension using *Macaulay 2* [9]. We find that $\alpha(G[H]) = \dim(R/I(G[H])) = 2 = n - \operatorname{pdim}(R/I(G[H]))$, so $G[H]$ is Cohen-Macaulay. However, $\alpha(H[G]) = \dim(R/I(H[G])) = 2 > n - \operatorname{pdim}(R/I(H[G])) = 1$, so $H[G]$ is not Cohen-Macaulay.

In light of the above example, we can ask what conditions on G and H allow us to conclude that the lexicographical product $G[H]$ is Cohen-Macaulay.

We end with a question concerning Lemma 3.3. Using *Macaulay 2* [9], we found that $\dim_k \tilde{H}_2(\Delta; k) = \frac{4d+3}{3}\binom{d-1}{2}$ for $d = 1, \ldots, 14$. This suggests that the inequality of Lemma 3.3 is actually an equality. We wonder if this is indeed true.

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